On the finiteness of Markov complexity of generalized Lawrence liftings

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Abstract

The notion of generalized Lawrence liftings and their Markov complexity for matrices $A \in \mathcal{M}_{d\times n}(\mathbb{Z})$ and $B \in \mathcal{M}_{p\times n}(\mathbb{Z})$ originated from Algebraic Statistics. We give necessary and sufficient conditions for the Markov complexity to be finite.

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1 Introduction

The notion of Markov basis originated from Algebraic Statistics. Let A be an element of $\mathcal{M}_{m\times n}(\mathbb{Z})$, for some positive integers m, n. The object of interest is the lattice $\mathcal{L}(A) := \mathrm{Ker}_{\mathbb{Z}}(A)$. A *Markov basis* of A is a finite subset $\mathcal{M}$ of $\mathcal{L}(A)$ such that whenever $\mathbf{w}, \mathbf{u} \in \mathbb{N}^n$ and $\mathbf{w} - \mathbf{u} \in \mathcal{L}(A)$ (i.e. $A\mathbf{w} = A\mathbf{u}$), there exists a subset $\{\mathbf{v}_i : i = 1, \ldots, s\}$ of $\mathcal{M}$ that *connects* $\mathbf{w}$ to $\mathbf{u}$. This means that for $1 \le p \le s$, $\mathbf{w} + \sum_{i=1}^{p} \mathbf{v}_i \in \mathbb{N}^n$ and $\mathbf{w} + \sum_{i=1}^{s} \mathbf{v}_i = \mathbf{u}$. A Markov basis $\mathcal{M}$ of A gives rise to a generating set of the lattice ideal

$$I_{\mathcal{L}(A)} := \langle x^{\mathbf{u}} - x^{\mathbf{v}} : \ A\mathbf{u} = A\mathbf{v} \rangle \ .$$

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Indeed each $\mathbf{u} \in \mathbb{Z}^n$ can be uniquely written as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ where $\mathbf{u}^+$, $\mathbf{u}^- \in \mathbb{N}^n$. It can be shown that if $\mathcal{M}$ is a Markov basis of A then the set $\{x^{\mathbf{u}^+} - x^{\mathbf{u}^-} : u \in \mathcal{M}\}$ is a generating set of $I_{\mathcal{L}(A)}$, see [4]. A Markov basis $\mathcal{M}$ of A is *minimal* if no subset of $\mathcal{M}$ is a Markov basis of A. It is possible that minimal Markov bases of A may have different cardinalities. The *universal Markov basis* of A, $\mathcal{M}(A)$, is the union of all minimal Markov bases of A of minimal cardinality, where we identify a vector $\mathbf{u}$ with $-\mathbf{u}$, see [3,10]. The *extended universal Markov basis* of A, $\mathcal{E}(A)$, is the union of all minimal Markov bases of A, where we identify a vector $\mathbf{u}$ with $-\mathbf{u}$. Note that the universal Markov basis is a subset of the extended universal Markov basis. When $\mathcal{L}(A)$ is a *positive lattice*, i.e. $\mathcal{L}(A) \cap \mathbb{N}^n = \{\mathbf{0}\}$, the graded Nakayama Lemma applies and all minimal Markov bases have the same cardinality. Thus, when $\mathcal{L}(A) \cap \mathbb{N}^n = \{\mathbf{0}\}$, the sets $\mathcal{M}(A)$ and $\mathcal{E}(A)$ are identical.

Another subset of $\mathcal{L}(A)$ that plays an important role in the study of lattice ideals, is the *Graver basis*, $\mathcal{G}(A)$, of A. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be nonzero integer vectors. We say that $\mathbf{u} = \mathbf{v} +_c \mathbf{w}$ is a *conformal decomposition* of $\mathbf{u}$ if $\mathbf{u}^+ = \mathbf{v}^+ + \mathbf{w}^+$ and $\mathbf{u}^- = \mathbf{v}^- + \mathbf{w}^-$. $\mathcal{G}(A)$ is the subset of $\mathcal{L}(A)$ whose elements have no conformal decomposition. It is always a finite set, see [11,6]. In this paper we show that $\mathcal{L}(A) \cap \mathbb{N}^n = \{\mathbf{0}\}$ is a necessary and sufficient condition for the inclusion $\mathcal{E}(A) \subset \mathcal{G}(A)$ to hold, see Theorem 2.1. We note that the sufficiency of this condition is well known and referred to, in the literature. However, since we could not find a written version of the proof, we give such a proof in Theorem 2.1 for completeness of this present exposition.

Hierarchical models in Algebraic Statistics encourage the study of *generalized Lawrence liftings* $\Lambda(A, B, r)$ for $r \geq 2$, where $A \in \mathcal{M}_{d\times n}(\mathbb{Z})$, $B \in \mathcal{M}_{p\times n}(\mathbb{Z})$, $r \in \mathbb{N}$:

$$\Lambda(A,B,r) = \begin{pmatrix} \overbrace{\begin{matrix} A & 0 & & 0 \end{matrix}}^{r-\text{times}} \\ \begin{matrix} 0 & A & & 0 \end{matrix} \\ \ddots \\ \begin{matrix} 0 & 0 & & A \end{matrix} \\ \begin{matrix} B & B & \cdots & B \end{matrix} \end{pmatrix},$$

see [10,8]. We denote the columns of A by $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and the columns of B by $\mathbf{b}_1, \ldots, \mathbf{b}_n$. The $(rd+p) \times rn$ matrix $\Lambda(A, B, r)$ has columns the vectors

$$\{\mathbf{a}_i \otimes \mathbf{e}_j \oplus \mathbf{b}_i : \ 1 \leq i \leq n, 1 \leq j \leq r\},$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ represents the canonical basis of $\mathbb{Z}^n$. When $B = I_n$ one gets the usual r–th *Lawrence lifting* $A^{(r)}$, see [10]. We note that $\mathcal{L}(\Lambda(A, B, r))$ is a sublattice of $\mathbb{Z}^{rn}$. Let $C \in \mathcal{L}(\Lambda(A, B, r))$. We can assign to C an $r \times n$ matrix $\mathcal{C}$ such that

$\mathcal{C}_{i,j} = C_{(i-1)n+j}$. Each row of $\mathcal{C}$ corresponds to an element of $\mathcal{L}(A)$ and the sum of the rows of $\mathcal{C}$ corresponds to an element in $\mathcal{L}(B)$. The number of nonzero rows of $\mathcal{C}$ is the *type* of C. The *complexity* of any subset of $\Lambda(A, B, r)$ is the largest type of any vector in that set. The *Markov complexity*, $m(A, B)$, is the largest type of any vector in the universal Markov basis of $\Lambda(A, B, r)$ as r varies. The *extended Markov complexity*, $e(A, B)$, is the largest type of any vector in the extended universal Markov basis of $\Lambda(A, B, r)$ as r varies. Certainly $e(A, B) \geq m(A, B)$. The *Graver complexity* $g(A, B)$ is the largest type of any vector in the Graver basis of $\Lambda(A, B, r)$ as r varies.

It is natural to ask whether and when any of the complexities $m(A, B)$, $e(A, B)$, $g(A, B)$ are finite. In [10] it was shown that for any matrix A, $g(A, I_n) < \infty$ and $e(A, I_n) \leq g(A, I_n)$. Moreover in [8] it was shown that if A is a matrix with positive integer entries, then $g(A, B) < \infty$ and $e(A, B) \leq g(A, B)$. We note that in both cases mentioned above, the lattices of the corresponding r-th (generalized) Lawrence lifting for $r \geq 2$, are positively graded. In this paper we show that if for any $r \geq 2$, $\mathcal{L}(\Lambda(A, B, r)) \cap \mathbb{N}^{rn} \neq \{\mathbf{0}\}$ then $m(A, B)$ and $e(A, B)$ are infinite, see Theorem 2.9. Thus $m(A, B)$ *is finite if and only if* $\Lambda(A, B, r)$ *is positively graded for any* $r \geq 2$. It follows that $e(A, B) = m(A, B)$, see Theorem 2.8. To find out whether $\mathcal{L}(\Lambda(A, B, r)) \cap \mathbb{N}^{rn} \neq \{\mathbf{0}\}$ for any $r \geq 2$, it suffices to check the intersection $\mathrm{Ker}_{\mathbb{Z}}(A) \cap \mathrm{Ker}_{\mathbb{Z}}(B)$, as it is shown in Lemma 2.4. We also prove in Theorem 2.5 that *if* $m(A, B) < \infty$ *and* $r \geq 2$ *then all minimal Markov bases of* $\Lambda(A, B, r)$ *have the same complexity, while this fails to be true if* $m(A, B) = \infty$. In the last section we give an explicit example where we show that if $m(A, B) = \infty$, then all is possible when considering the complexities of individual minimal Markov bases of $\Lambda(A, B, r)$. In this example in particular, one can find minimal Markov bases with complexities ranging from 1 to r.

2 On the finiteness of Markov complexity

Let $D \in \mathcal{M}_{m \times n}(\mathbb{Z})$. We let $\mathcal{L} := \mathcal{L}(D) \subset \mathbb{Z}^n$ and $\mathcal{L}_{pure}$ be the sublattice of $\mathcal{L}$ generated by the elements in $\mathcal{L} \cap \mathbb{N}^n$. This is the *pure sublattice* of $\mathcal{L}$, see [3]. In [3, Theorem 4.18], it was shown that

if $\mathrm{rank}(\mathcal{L}_{pure}) > 1$ *or* $\mathrm{rank}(\mathcal{L}_{pure}) = 1$ *and* $\mathcal{L} \neq \mathcal{L}_{pure}$, *then the universal Markov basis of* D, $\mathcal{M}(D)$, *is infinite.*

It is automatic that in these cases, $\mathcal{E}(D)$, the extended universal Markov basis of D, is infinite.

Suppose now that $\mathrm{rank}(\mathcal{L}_{pure}) = 1$ and $\mathcal{L} = \mathcal{L}_{pure}$. We let $\mathbf{0} \neq \mathbf{w} \in \mathbb{N}^n$ be such that $\mathcal{L} = \langle \mathbf{w} \rangle$. It is immediate that $\mathbf{w} \in \mathcal{G}(D)$ and thus $\mathcal{M}(D) \subset \mathcal{G}(D)$. On the other hand, one can easily see that $\{k\mathbf{w}, l\mathbf{w}\}$ is a minimal Markov basis of D for any two

relatively prime integers $k, l \geq 2$. Thus $\mathcal{E}(D)$ is infinite. Since $\mathcal{G}(D)$ is a finite set, it follows that

if $\operatorname{rank}(\mathcal{L}_{pure}) > 1$ *or* $\operatorname{rank}(\mathcal{L}_{pure}) = 1$ *and* $\mathcal{L} \neq \mathcal{L}_{pure}$*, then the universal Markov basis of D is not contained in the Graver basis of D. If* $\operatorname{rank}(\mathcal{L}_{pure}) \geq 1$ *then the extended universal Markov basis of D is not contained in the Graver basis of D.*

Let $\mathbf{u} \in \mathcal{L}$ and consider the set $\mathcal{F}(\mathbf{u}^+) := \{\mathbf{t} \in \mathbb{N}^n : \mathbf{u}^+ - \mathbf{t} \in \mathcal{L}\}$. This is a finite set if and only if $\mathcal{L} \cap \mathbb{N}^n = \{\mathbf{0}\}$. We join two elements $\mathbf{w_1}$, $\mathbf{w_2}$ of $\mathcal{F}(\mathbf{u}^+)$ by an edge if and only if there is $\mathbf{v} \in \mathcal{L}$ such that $\mathbf{v}^+$ is componentwise smaller than $\mathbf{w_1}$ and $\mathbf{w_2}$, meaning that at least one component of the nonnegative vector $\mathbf{w_i} - \mathbf{v}^+$ is strictly positive, for $i = 1, 2$. We let $\mathcal{G}_\mathbf{u}$ be the graph thus produced. When $\mathcal{L} \cap \mathbb{N}^n = \{\mathbf{0}\}$, a necessary condition for $\mathbf{u} \in \mathcal{L}$ to be in $\mathcal{M}(D)$, is the following criterion that appears in [2, Theorem 2.9], [5, Theorem 1.3.2] and [3, Theorem 3.13].

If $\mathcal{L} \cap \mathbb{N}^n = \{\mathbf{0}\}$ and $\mathbf{u}$ is in $\mathcal{M}(D)$ then $\mathbf{u}^+$ and $\mathbf{u}^-$ belong to different connected components of $\mathcal{G}_\mathbf{u}$.

We will use this criterion in the proof of the next theorem.

Theorem 2.1 *The extended universal Markov basis of D is contained in the Graver basis of D if and only if $\mathcal{L}(D) \cap \mathbb{N}^n = \{\mathbf{0}\}$. The universal Markov basis of D is contained in the Graver basis of D if and only if $\mathcal{L}(D) \cap \mathbb{N}^n = \{\mathbf{0}\}$ or $\mathcal{L}(D) = \mathcal{L}(D)_{pure}$ and* $\operatorname{rank} \mathcal{L}(D) = 1$*.*

Proof. Let $\mathcal{L} := \mathcal{L}(D)$. By the remarks preceding the theorem it remains to be shown that if $\mathcal{L} \cap \mathbb{N}^n = \{\mathbf{0}\}$, $\mathbf{u} \in \mathcal{L}$, $\mathbf{u} \notin \mathcal{G}(D)$ then $\mathbf{u} \notin \mathcal{M}(D)$. Since $\mathbf{u} \notin \mathcal{G}(D)$ there exist nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathcal{L}$ such that $\mathbf{u} = \mathbf{v} +_c \mathbf{w}$. Thus $\mathbf{u}^+ = \mathbf{v}^+ + \mathbf{w}^+$ and $\mathbf{u}^- = \mathbf{v}^- + \mathbf{w}^-$. It follows that $\mathbf{u}^+$, $\mathbf{u}^-$ and $\mathbf{u}^+ - \mathbf{v} = \mathbf{w}^+ + \mathbf{v}^-$ are all in $\mathcal{F}(\mathbf{u}^+)$. Next we show that $\mathbf{v}^+$ is nonzero. Indeed suppose not. Since $\mathbf{v}^- = \mathbf{v} - \mathbf{v}^+ = \mathbf{v} \in \mathcal{L} \cap \mathbb{N}^n = \{\mathbf{0}\}$, it follows that $\mathbf{v} = \mathbf{0}$, a contradiction. Similarly $\mathbf{w}^+, \mathbf{v}^-, \mathbf{w}^-$ are nonzero. Thus in the fiber $\mathcal{F}(\mathbf{u}^+)$, the elements $\mathbf{u}^+$, $\mathbf{w}^+ + \mathbf{v}^-$ are connected by an edge because of $\mathbf{w}$ and similarly $\mathbf{u}^-$, $\mathbf{w}^+ + \mathbf{v}^-$ are connected by an edge because of $-\mathbf{v}$. It follows that $\mathbf{u}^+$, $\mathbf{u}^-$ belong to the same connected component of $\mathcal{G}_\mathbf{u}$ and thus $\mathbf{u} \notin \mathcal{M}(D)$. $\square$

Let " $>$ " be any monomial order. We briefly note that if $x^{\mathbf{u}^+} - x^{\mathbf{u}^-}$ is in the reduced Gröbner basis of $I_{\mathcal{L}(D)}$, then $\mathbf{u}$ is in the Graver basis of D, see [11]. The universal Gröbner basis of D consists of all $\mathbf{u}$ such that $x^{\mathbf{u}^+} - x^{\mathbf{u}^-}$ is in a reduced Gröbner basis of $I_{\mathcal{L}(D)}$ and is a finite set. Thus, if $\mathcal{L}(D) \cap \mathbb{N}^n \neq \{\mathbf{0}\}$ then the universal Markov basis of D cannot be contained in the universal Gröbner basis of D. We note that even when $\mathcal{L}(D) \cap \mathbb{N}^n = \{\mathbf{0}\}$, the containment does not hold, as the next example shows.

Example 2.2 Let

$$D = \begin{pmatrix} 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 4 & 0 & 4 & 0 & 3 & 3 & 3 & 3 \\ 4 & 0 & 0 & 4 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 6 & 0 & 6 & 0 \\ 2 & 2 & 2 & 2 & 6 & 0 & 0 & 6 \end{pmatrix}.$$

The set $\{(1,1,-1,-1,0,0,0,0),(0,0,0,0,1,1,-1,-1),(2,2,1,1,-1,-1,-1,-1)\}$ is a minimal Markov basis of D. It follows from [4] that

$$I_D = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8, x_1^2x_2^2x_3x_4 - x_5x_6x_7x_8).$$

We note that for any monomial order, the initial term of $x_1x_2 - x_3x_4$ divides $x_1^2x_2^2x_3x_4$ while the initial term of $x_5x_6 - x_7x_8$ divides $x_5x_6x_7x_8$. Therefore the element $x_1^2x_2^2x_3x_4 - x_5x_6x_7x_8$ does not belong to a reduced Gröbner basis of $I_{\mathcal{L}(D)}$ and the vector $(2,2,1,1,-1,-1,-1,-1)$ is not in the universal Gröbner basis of D. For a different proof based on the geometry of the fibers one can use the arguments of [11, Chapter 7]. Note that D has 12 different Markov bases. Of those bases, exactly 4 are subsets of the universal Gröbner basis of D.

We say that $\mathbf{u}$ is $\mathcal{L}(D)$*–primitive* if $\mathbf{u} \neq \mathbf{0}$ and $\mathbb{Q}\mathbf{u} \cap \mathcal{L}(D) = \mathbb{Z}\mathbf{u}$. For $\mathbf{u} \in \mathbb{Z}^n$, we let $\mathrm{supp}(\mathbf{u}) := \{i : \ u_i \neq 0\}$. For $X \subset \mathcal{L}(D)$, we let

$$\mathrm{supp}(X) := \bigcup_{\mathbf{u} \in X} \mathrm{supp}(\mathbf{u}) \ .$$

Suppose that $\mathcal{L}(D) \cap \mathbb{N}^n \neq \{\mathbf{0}\}$. In [3] it was shown that there exists an $\mathcal{L}(D)$*–primitive* element $\mathbf{u} \in \mathcal{L}(D) \cap \mathbb{N}^n$ such that $\mathrm{supp}(\mathbf{u}) = \mathrm{supp}(\mathcal{L}(D)_{pure})$, [3, Proposition 2.7, Proposition 2.10]. This element can be extended to a minimal basis of $\mathcal{L}(D)_{pure}$ and then to a minimal Markov basis of D of minimal cardinality by [3, Theorem 2.12, Theorem 4.1, Theorem 4.11]. This is the point of the next lemma.

Lemma 2.3 *If $\mathcal{L}(D)_{pure} \neq \{\mathbf{0}\}$, there exists an $\mathcal{L}(D)$–primitive element $\mathbf{v} \in \mathbb{N}^n$ and a minimal Markov basis of D of minimal cardinality that contains $\mathbf{v}$, such that* $\mathrm{supp}(\mathbf{v}) = \mathrm{supp}(\mathcal{L}(D)_{pure})$.

Next we consider generalized Lawrence liftings, for $A \in \mathcal{M}_{d \times n}(\mathbb{Z})$, $B \in \mathcal{M}_{p \times n}(\mathbb{Z})$ and $2 \leq r \in \mathbb{N}$. We let

$$\mathcal{L}_r := \mathcal{L}(\Lambda(A,B,r)), \quad \mathcal{L}_{A,B} := \mathrm{Ker}_{\mathbb{Z}}(A) \cap \mathrm{Ker}_{\mathbb{Z}}(B) \ .$$

We note that $\mathcal{L}_r \subset \mathbb{Z}^{rn}$ while $\mathcal{L}_{A,B} \subset \mathbb{Z}^n$.

Proposition 2.4 *$\mathcal{L}_{A,B} \cap \mathbb{N}^n \neq \{\mathbf{0}\}$ if and only if $\mathcal{L}_r \cap \mathbb{N}^{rn} \neq \{\mathbf{0}\}$ for any $r \geq 2$.*

Proof. Let $C \in \mathcal{L}_{A,B} \cap \mathbb{N}^n$. We think of the elements of $\mathcal{L}_r$ as $r \times n$ matrices, as explained in the introduction. We have that $[C \cdots C]^T \in \mathcal{L}_r \cap \mathbb{N}^{rn}$. Conversely, if $[C_1 \cdots C_r]^T \in \mathcal{L}_r \cap \mathbb{N}^{rn}$ then $C_1 + \cdots + C_r \in \mathcal{L}_{A,B} \cap \mathbb{N}^n$. □

Suppose that $\mathcal{L}_r \cap \mathbb{N}^{rn} = \{\mathbf{0}\}$. Let $U \in \mathcal{L}_r$ and let $\mathcal{U}$ the corresponding $r \times n$ matrix with $\mathbf{u}_i$ as its i-th row. We define $\sigma(\mathcal{U}) = \{i : \ \mathbf{u}_i \neq 0, \ 1 \leq i \leq r\}$. Thus the type of $\mathcal{U}$ is the cardinality of $\sigma(\mathcal{U})$. The $\Lambda(A, B, r)$–*degree* of U is the vector $\Lambda(A, B, r)U^+$. Thus the $\Lambda(A, B, r)$–degree of U is in the span $\mathbb{N}(\mathbf{a}_i \otimes \mathbf{e}_j \oplus \mathbf{b}_i : \ 1 \leq i \leq n, j \in \sigma(\mathcal{U}))$. It is well known that the $\Lambda(A, B, r)$–degrees of any minimal Markov basis of $\Lambda(A, B, r)$ of minimal cardinality are invariants of $\Lambda(A, B, r)$, see [11].

Theorem 2.5 *Let $\mathcal{L}_r \cap \mathbb{N}^{rn} = \{\mathbf{0}\}$. The complexity of a minimal Markov basis of $\Lambda(A, B, r)$ is an invariant of $\Lambda(A, B, r)$.*

Proof. Let M_1, M_2 be two minimal Markov bases of $I_{\mathcal{L}_r}$. It is enough to show that the complexity of M_1 is less than or equal to the complexity of M_2. Let $\mathcal{U} = [\mathbf{u}_1 \cdots \ \mathbf{u}_r]^T \in M_1$ be such that the type of $\mathcal{U}$ is equal to the complexity of M_1. We let $\mathcal{V} = [\mathbf{v}_1 \cdots \ \mathbf{v}_r]^T \in M_2$ be such that the $\Lambda(A, B, r)$-degree of V is the same as the $\Lambda(A, B, r)$-degree of U. Thus the $\Lambda(A, B, r)$–degree of V is in $\mathbb{N}(\mathbf{a_i} \otimes \mathbf{e_j} \oplus \mathbf{b_i} : \ 1 \leq i \leq n, j \in \sigma(\mathcal{U}))$. This implies that $\mathbf{v}_i^+ = 0$ for every $i \notin \sigma(\mathcal{U})$. Since every nonzero element in $\mathrm{Ker}_{\mathbb{Z}}(A)$ has a nonzero positive part (and a nonzero negative part) it follows that $\mathbf{v}_i = 0$ for every $i \notin \sigma(\mathcal{U})$. Thus $\sigma(\mathcal{V}) \subset \sigma(\mathcal{U})$. Reversing the argument we get that $\sigma(\mathcal{U}) = \sigma(\mathcal{V})$. Therefore the complexity of M_1 is less than or equal to the complexity of M_2. □

As in [8, Theorem 3.5] one can prove the following statement for arbitrary integer matrices $A \in \mathcal{M}_{d \times n}(\mathbb{Z})$, $B \in \mathcal{M}_{p \times n}(\mathbb{Z})$.

Theorem 2.6 *The Graver complexity $g(A, B)$ is the maximum 1–norm of any element in the Graver basis $\mathcal{G}(B \cdot \mathcal{G}(A))$. In particular, we have $g(A, B) < \infty$.*

Suppose that $\mathcal{L}_r \cap \mathbb{N}^{rn} \neq \{\mathbf{0}\}$. Next we show that $\Lambda(A, B, r)$ has a minimal Markov basis (of minimal cardinality) whose complexity is r.

Lemma 2.7 *Suppose that $\mathcal{L}_r \cap \mathbb{N}^{rn} \neq \{\mathbf{0}\}$. There exists a minimal Markov basis of $\Lambda(A, B, r)$ of minimal cardinality, that contains an element of type r.*

Proof. We first show that $\mathcal{L}_r \cap \mathbb{N}^{rn}$ has an element of type r. By Lemma 2.4, $\mathcal{L}_{A,B} \cap \mathbb{N}^n \neq \{\mathbf{0}\}$. We let $\mathbf{w} \in \mathcal{L}_{A,B} \cap \mathbb{N}^n$ be such that $\mathrm{supp}(\mathbf{w}) = \mathrm{supp}((\mathcal{L}_{A,B})_{pure})$. It follows that

$$\begin{pmatrix} \mathbf{w} \\ \vdots \\ \mathbf{w} \end{pmatrix} \in \mathcal{L}_r \cap \mathbb{N}^{rn}$$

has type r. Since $(\mathcal{L}_r)_{pure} = \langle \mathcal{L}_r \cap \mathbb{N}^{rn} \rangle$, we are done by Lemma 2.3. $\square$

The next theorem is an immediate consequence of Lemma 2.7.

Theorem 2.8 *The Markov complexity $m(A,B)$ is equal to the extended Markov complexity $e(A,B)$ for all $A \in \mathcal{M}_{d\times n}(\mathbb{Z})$, $B \in \mathcal{M}_{p\times n}(\mathbb{Z})$.*

Proof. If $\mathcal{L}_{A,B} \cap \mathbb{N}^n \neq \{\mathbf{0}\}$ then $m(A,B)$ and $e(A,B)$ are both infinite by Proposition 2.4 and Lemma 2.7. If $\mathcal{L}_{A,B} \cap \mathbb{N}^n = \{\mathbf{0}\}$ then $e(A,B) = m(A,B)$ by Nakayama's lemma. $\square$

Next we completely characterize the cases where $m(A,B) < \infty$.

Theorem 2.9 *The following are equivalent:*

(1) the Markov complexity $m(A,B)$ is finite,
(2) $\mathcal{L}_{A,B}$ is a positive lattice,
(3) $\forall r \geq 2$, all minimal Markov bases of $\Lambda(A,B,r)$ have the same complexity.

Proof. For (1)$\Leftrightarrow$(2) we only have to show that if $\mathcal{L}_{A,B} \cap \mathbb{N}^n = \{\mathbf{0}\}$ then $m(A,B)$ is finite. By Lemma 2.4, $\mathcal{L}_r \cap \mathbb{N}^{rn} = \{\mathbf{0}\}$ for all $r \geq 2$. By Theorem 2.1 the universal Markov basis of $\Lambda(A,B,r)$ is a subset of the Graver basis of $\Lambda(A,B,r)$ for any r. Thus $m(A,B) \leq g(A,B)$. The latter one is finite, by Theorem 2.6.

We note that Theorem 2.5 gives the implication (1)$\Rightarrow$ (3). For the reverse implication, let $r \geq g(A,B)$ and assume to the contrary that $m(A,B) = \infty$. By Lemma 2.7, there is a minimal Markov basis $\mathcal{M}_1$ of $\Lambda(A,B,r)$ of complexity r. We let $\mathcal{M}_2$ be the universal Gröbner basis of D. We note that $\mathcal{M}_2$ is a Markov basis of D and is contained in the Graver basis. Thus the complexity of $\mathcal{M}_2$ is less than or equal to $g(A,B) < r$. The Markov bases $\mathcal{M}_1, \mathcal{M}_2$ have different complexities, a contradiction. $\square$

Remark 2.10 Suppose that $r \geq 2$ and $\mathcal{L}_r \cap \mathbb{N}^{rn} \neq \{\mathbf{0}\}$. Let $\mathbf{v}$ be the $\mathcal{L}_r$-primitive element of Lemma 2.3. By adding positive multiples of $\mathbf{v}$ to the other elements of the Markov basis of Lemma 2.3 the new set is still a minimal Markov basis of $\Lambda(A,B,r)$, see [3], with the property that all of its elements are of type r.

3 Example

In this section we give an example of matrices A, B such that for any given $r \geq 2$, $\Lambda(A,B,r)$ has a minimal Markov basis with elements of type 1 and 2, a minimal Markov basis with elements of any type from 1 to r, a minimal Markov basis with

all elements of type r and an infinite universal Markov basis.

We let $A_1 \in \mathcal{M}_{2\times m}(\mathbb{Z})$, $A_2 \in \mathcal{M}_{2\times 2}(\mathbb{Z})$, $A \in \mathcal{M}_{2\times(m+2)}(\mathbb{Z})$, $B_2 \in \mathcal{M}_{m\times 2}(\mathbb{Z})$ and $B \in \mathcal{M}_{m\times(m+2)}(\mathbb{Z})$ be the following matrices:

$$A_1 = \begin{pmatrix} 1 \cdots 1 \\ 0 \cdots 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \; A = (A_1|A_2)\,, \quad B_2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \; B = (I_m|B_2)\ .$$

We consider the matrix $\Lambda(A, B, r)$. After column permutations it follows that

$$\Lambda(A,B,r) = \begin{pmatrix} A & 0 & & 0 \\ 0 & A & & 0 \\ & & \ddots & \\ 0 & 0 & & A \\ B & B & \cdots & B \end{pmatrix} \to \left(\begin{array}{cccc|cccc} A_1 & 0 & & 0 & A_2 & 0 & & 0 \\ 0 & A_1 & & 0 & 0 & A_2 & & 0 \\ & & \ddots & & & & \ddots & \\ 0 & 0 & & A_1 & 0 & 0 & & A_2 \\ I_m & I_m & \cdots & I_m & B_2 & B_2 & \cdots & B_2 \end{array}\right)$$

We note that the lattice $\mathcal{L}\left(\Lambda(A_1, I_m, r)|\Lambda(A_2, B_2, r)\right)$ is isomorphic to the direct sum of the lattices $\mathcal{L}(\Lambda(A_1, I_m, r))$ and $\mathcal{L}(\Lambda(A_2, B_2, r))$ and thus there is a one to one correspondence between the Markov bases of $\Lambda(A, B, r)$ and unions of the Markov bases of $\mathcal{L}(\Lambda(A_1, I_m, r))$ and $\mathcal{L}(\Lambda(A_2, B_2, r))$.

The matrix $\Lambda(A_1, I_m, r)$ is the defining matrix of the toric ideal of the complete bipartite graph $K_{m,r}$ and has a unique Markov basis corresponding to cycles of length 4: all its elements have type 2, see [9] and [10, Example 5]. We denote by C_i the columns of $\Lambda(A_2, B_2, r)$, for $i = 1, \ldots, 2r$. We note that $C_1, C_3, \ldots, C_{2r-1}$ are linearly independent while $C_{2l-1} = -C_{2l}$ for $1 \le l \le r$. It follows that the lattice $\mathcal{L}(\Lambda(A_2, B_2, r))$ has rank r and is pure. Thus it has infinitely many Markov bases, see [3]. We consider the following Markov basis of $\Lambda(A_2, B_2, r)$ consisting of elements of type 1:

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 1 & 1 \end{pmatrix} \right\}.$$

For fixed $1 \leq a \leq r$ and $1 \leq b \leq m+2$ we let $E_{a,b}$ be the matrix of $\mathcal{M}_{r\times(m+2)}(\mathbb{Z})$ which has 1 on the (a,b)-th entry and 0 everywhere else. Moroever for $1 \leq i < j \leq r$, $1 \leq k < l \leq m$ and $1 \leq s \leq r$, we let $E_{(i,j),(k,l)} \in \mathcal{M}_{r\times(m+2)}(\mathbb{Z})$ and $T_s \in \mathcal{M}_{r\times(m+2)}(\mathbb{Z})$ be the matrices

$$E_{(i,j),(k,l)} = E_{i,k} - E_{i,l} - E_{j,k} + E_{j,l}, \quad T_s = E_{s,m+1} + E_{s,m+2} \ .$$

It follows that the set $\mathcal{M} = \{T_1, \ldots, T_r\} \cup \{E_{(i,j),(k,l)} : \ 1 \leq i < j \leq r, 1 \leq k < l \leq m\}$ is a minimal Markov basis of $\Lambda(A, B, r)$ of cardinality $r + \binom{r}{2}\binom{m}{2}$. The elements of $\mathcal{M}$ have type 1 and 2.

Note that the set

$$\{T_1, T_1 + T_2, \ldots, T_1 + \cdots + T_r\} \cup \{E_{(i,j),(k,l)} : \ 1 \leq i < j \leq r, 1 \leq k < l \leq m\}$$

is a minimal Markov basis of $\Lambda(A, B, r)$ and the type of its elements range from 1 to r. Moreover if $T = \sum_{s=1}^{r} T_s$, then the set

$$\{T, T + T_2, \ldots, T + T_r\} \cup \{T + E_{(i,j),(k,l)} : \ 1 \leq i < j \leq r, 1 \leq k < l \leq m\}$$

is a minimal Markov basis of $\Lambda(A, B, r)$ such that all its elements are of type r, see [3].

We remark that if S is any integer linear combination of the elements T_s, $1 \leq s \leq r$ and $1 \leq i < j \leq r, 1 \leq k < l \leq m$ then the element $S + E_{(i,j),(k,l)}$ belongs to the infinite universal Markov basis of $\Lambda(A, B, r)$.

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